Exact Equations

Def: A first order ODE of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is <u>exact</u> if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

If the equation is exact then there must exist a function $F(x,y)$ such that

$$dF = M(x,y)dx + N(x,y)dy$$

because we know that

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

and we know that the mixed 2nd partials are equal, that is

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}.$$

So if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then it must be that $M(x,y) = \frac{\partial F}{\partial x}$, $N(x,y) = \frac{\partial F}{\partial y}$

for some function $F(x,y)$ and

$$dF = M(x,y)dx + N(x,y)dy = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0$$

and $\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}$.

Thus if we can find a function $F(x,y)$ such that $dF = Mdx + Ndy = 0$ then the function $F = C$ satisfies the given equation.

To find $F(x,y)$ we first verify that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. This means that F exists and $M = \frac{\partial F}{\partial x}$, $N = \frac{\partial F}{\partial y}$. We may begin with either of these relationships to find F. Suppose we choose

$$\frac{\partial F}{\partial x} = M(x,y)$$

Then $F = \int M(x,y)dx + G(y)$

To find $G(y)$ we then differentiate —

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)dx + G'(y) = N(x,y)$$

which allows us to solve for $G(y)$ and we now have $F(x,y)$. Since $dF = 0$ we know that $F(x,y) = c$ satisfies the given equation, $Mdx + Ndy = 0$.

Ex: Find the general solution of

$$2xy\,dx + (1+x^2)\,dy = 0$$

First, check exactness –

$$\frac{\partial}{\partial y}(2xy) = 2x$$

and $\frac{\partial}{\partial x}(1+x^2) = 2x$

Since $\frac{\partial}{\partial y}(2xy) = \frac{\partial}{\partial x}(1+x^2)$ then there exists a function $F(x,y)$ such that $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 2xy\,dx + (1+x^2)\,dy = 0$

Since $\frac{\partial F}{\partial x} = 2xy$ then $F(x,y) = \int 2xy\,dx = x^2y + G(y)$ where $G(y)$ is a function of y to be determined. To find $G(y)$ we note that $\frac{\partial F}{\partial y} = x^2 + G'(y) = 1 + x^2$

So $G'(y) = 1$ and $G(y) = y$

Thus $F(x,y) = x^2y + G(y)$

$$= x^2y + y$$

and because we know $dF = 0$, it must be

that $F(x,y) = C$ or

$$\boxed{x^2y + y = c}$$

is the general solution.

The nice thing about differential equations is that you can verify your solution. Rewriting the original equation as

$\frac{dy}{dx} = \frac{-2xy}{1+x^2}$ you can check your solution –

if $x^2y + y = c$

then $y = \frac{c}{1+x^2}$

and $\frac{dy}{dx} = \frac{-c}{(1+x^2)^2}(2x)$

$$= \frac{-2x}{1+x^2} \cdot \frac{c}{1+x^2}$$

$$= \frac{-2xy}{1+x^2} \quad \checkmark$$

and the solution does indeed satisfy the given equation.

Notice that in the previous example there was nothing special about the order we chose. We could have started by integrating $N(x,y)$. Sometimes one way is easier. That is, if –

$$2xy\,dx + (1+x^2)\,dy = 0$$

$$\frac{\partial}{\partial y}(2xy) = \frac{\partial}{\partial x}(1+x^2) = 2x$$

So the equation is exact and there exists $F(x,y) = c$ such that

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 2xy\,dx + (1+x^2)\,dy = 0$$

So $\frac{\partial F}{\partial y} = (1+x^2)$

$$F = \int (1+x^2)\,dy$$

$$= (1+x^2)y + G(x)$$

$$\frac{\partial F}{\partial x} = 2xy + G'(x) = 2xy$$

So $G'(x) = 0$, $G(x) = C_1 = \text{constant}$,

and $F(x,y) = (1+x^2)y + C_1 = C$

or $\boxed{(1+x^2)y = C}$ is the general solution.

This is, of course, the same answer!

Ex: Find the general solution of $y' = \dfrac{2+ye^{xy}}{2y - xe^{xy}}$.

First we write the equation in differential form –

$$(2+ye^{xy})\,dx + (xe^{xy} - 2y)\,dy = 0$$

Then apply the test for exactness –

$$\frac{\partial}{\partial y}(2+ye^{xy}) = e^{xy} + xye^{xy} = \frac{\partial}{\partial x}(xe^{xy} - 2y)$$

So the equation is exact and there exists a function $F(x,y) = c$ such that

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = (2+ye^{xy})\,dx + (xe^{xy} - 2y)\,dy = 0$$

So $\frac{\partial F}{\partial x} = (2+ye^{xy})$

$$F = 2x + e^{xy} + G(y)$$

$$\frac{\partial F}{\partial y} = xe^{xy} + G'(y)$$

But since $\frac{\partial F}{\partial y} = xe^{xy} - 2y$ it must be that

$$G'(y) = -2y$$
$$G(y) = -y^2$$

and finally

$$F(x,y) = 2x + e^{xy} - y^2$$

Since $F(x,y) = c$ satisfies the given equation we have

$\boxed{2x + e^{xy} - y^2 = c}$ is the general solution to the given differential equation. You should verify that indeed it is a solution but this time you have to differentiate implicitly.

If $2x + e^{xy} - y^2 = c$ then

$$2 + e^{xy}\left(y + x\frac{dy}{dx}\right) - 2y\frac{dy}{dx} = 0$$

or $$(xe^{xy} - 2y)\,dy + (2 + ye^{xy})\,dx = 0$$ ✓

which is indeed the equation we wanted to solve!